# HSC Mathematics Assessment Task 2

**7 March 2007**

**Topics:** **The Second Derivative and Geometrical Applications of Calculus; Series and Applications of Series**

Time Allowed: 70 minutes + 2 minutes reading time

Instructions: Start each question on a new page.
Show all necessary working, writing on one side of the paper only.
Work down the page and do not work in columns.
Leave a margin on the left hand side of the page.
Marks may not be awarded for untidy or poorly arranged work.

Name:____________________________________ Class:__________.

| | | | | | | |
|---|---|---|---|---|---|---|
| H5 | H7 | H5 | H5 | H5 | | % |
| | H6 | H6 | H4 | H4 | | |
| 12 | 14 | 13 | 11 | 13 | 63 | |

| | |
|---|---|
| H4 | |
| H5 | |
| H6 | |
| H7 | |

**Question 1:** (12 marks) **Marks**

(a) For the sequence 1, 4, 7, 10, . . .

(i) Find the common difference. **1**

(ii) Find the value of the 50th term. **2**

(b) (i) $\sum_{r=1}^{n}(60-4r)=360$. Find $n$. **3**

(ii) Explain the significance of your answer. **1**

(c) For a certain geometric series, the third term is 12 and the sixth term is 324.

(i) Find the common ratio. **2**

(ii) Find the sum of the first 8 terms. **3**

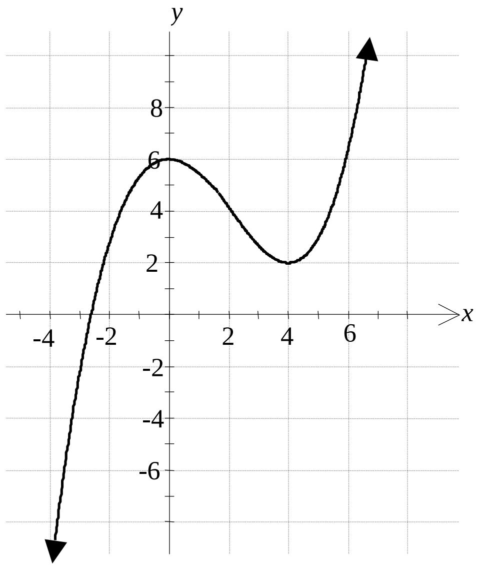
**Question 2:** (14 marks) **Marks**

(a) The volume, $V$, of water in a dam at time, $t$, was monitored over a period of time. When monitoring began, the dam was 60% full. During the monitoring period, the volume decreased however, government measures to arrest this decrease were proving effective.

(i) Sketch the graph of $V$ against $t$. **1**

(ii) What can be concluded about $\frac{dV}{dt}$ and $\frac{d^2V}{dt^2}$? **1**

(b) The diagram illustrates the graph of $y = f(x)$. There is a point of inflexion at $x = 2$.




(i) For what value(s) of $x$ is

(α) $y = f(x)$ increasing **1**

(β) $y = f(x)$ concave down **1**

(γ) $f'(x) = 0$ **1**

(δ) $f''(x) = 0$ **1**

(ε) $f'(x) < 0$ and $f''(x) > 0$ **2**

(ii) What is the minimum value of $y = f(x)$ in the domain $-3 \leq x \leq 6$? **1**

(c) For what values of $x$ is the graph of $y = x^3 + 3x^2 - 9x - 22$ concave up? **2**

(d) Find $f''(2)$ for the function $f(x) = (x^2 - 3)^4$. **3**

**Question 3:** (13 marks)

(a) For the graph of $y = 4x^3 - x^4$

(i) At what point(s) does the graph cut the $x$-axis? **1**

(ii) Find any stationary points and determine their nature. **5**

(iii) Find any points of inflexion. **2**

(iv) Sketch the graph, showing these details. **2**

(b) For a particular curve $\frac{d^2y}{dx^2} = 6x - 4$. The curve has a stationary point at (1, 12).
Find the equation of the curve. **3**

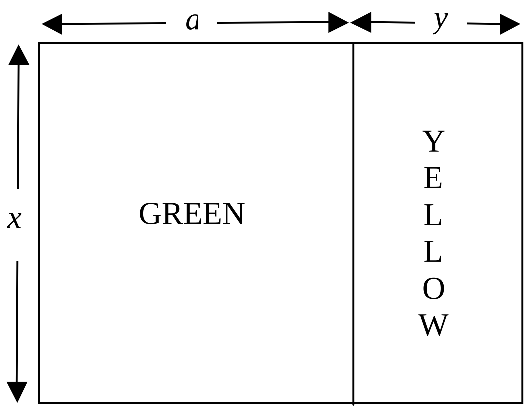
**Question 4:** (11 marks) **Marks**

(a) Find a primitive function of each of the following:

(i) $\dfrac{8+x^2}{x^2}$ **2**

(ii) $4\sqrt[3]{x}$ **1**

(b) The diagram below is design for a new Australian flag. The flag consists of two rectangular regions, one yellow and one green. The perimeter of the entire flag is 376 cm and the green region covers an area of 6561 cm$^2$.




All lengths are in centimetres.

Diagram not to scale.

Let the width of the yellow region be $y$ cm and the length and width of the green region be $x$ cm and $a$ cm respectively as shown in the diagram.

(i) Show that $a = 188 - x - y$. **1**

(ii) Hence show that the width of the yellow vertical stripe is given by

$y = 188 - x - \dfrac{6561}{x}$ cm. **2**

(iii) Find the dimensions of this flag if the width of the yellow stripe is to be a maximum. **5**

**Question 5:** (13 marks)

(a) Consider the series $6(x-1)+12(x-1)^2+24(x-1)^3+\ldots$

(i) For what values of $x$ will the series have a limiting sum? **3**

(ii) Find the limiting sum when $x = 1{\cdot}25$. **2**

**Question 5 continued:** **Marks**

(b) A benefactor donated \$12 000 to a school to be used to award a scholarship of \$1000 to a deserving student each year. The school invests the money at the beginning of the year in an account which pays interest at 6% pa compounded monthly and awards the scholarship at the end of the year, immediately after the interest has been paid. Let $A_n$ represent the amount remaining in the account immediately after $n$ scholarships have been awarded.

(i) Show that the amount of money remaining in the account immediately after the second scholarship is awarded is given by **2**

$$A_2 = 12000(1\cdot 005)^{24} - 1000\left[1 + 1\cdot 005^{12}\right]$$

(ii) Show that the amount remaining in the account immediately after $n$ scholarships are awarded is given by the expression **3**

$$A_n = 12000(1.005)^{12n} - 1000\left[\frac{(1.005)^{12n} - 1}{(1.005)^{12} - 1}\right]$$

(iii) For how many years can the full scholarship be awarded? **2**

(iv) In the year following the award of the last *full* scholarship, the remaining amount is awarded. What is the amount of this final scholarship? **1**

END OF TEST

# Mathematics Assessment Task 2: 2007 ~ Solutions.



## Question 1:

a) i) $d = 10-7 = 7-4 = 4-1 = 3$

$\therefore$ the difference is 3

ii) $a=1$, $d=3$, $n=50$

$T_n = a + (n-1)d$

$\therefore T_{50} = 1 + (50-1)(3)$

$= 148$

b) i) $\sum_{r=1}^{n} (60-4r) = 360$

$\therefore 56 + 52 + 48 + \ldots + (60-4n) = 360$

$a = 56$, $d = -4$, $S_n = 360$, $n = ?$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\therefore \quad 360 = \frac{n}{2}\left[2(56) + (n-1)(-4)\right]$

$360 = \frac{n}{2}\left[112 - 4n + 4\right]$

$360 = \frac{n}{2}\left[116 - 4n\right]$

$360 = 56n - 2n^2$

$n^2 - 28n + 180 = 0$

$(n-10)(n-18) = 0$

$n = 10, 18$

ii) As both $S_{10}$ and $S_{18}$ are equal, the terms from $T_{11}$ to $T_{18}$ must have a sum of zero.

c) i) $T_3 = 12 \quad \therefore ar^2 = 12 \ \ldots\ (1)$

$T_6 = 324 \quad \therefore ar^5 = 324 \ \ldots\ (2)$

$\frac{(2)}{(1)}: \quad \frac{ar^5}{ar^2} = \frac{324}{12}$

$r^3 = 27$

$r = 3$

ii) sub into (1): $a(3)^2 = 12$

$\therefore \quad a = \frac{12}{9}$

$= 1\frac{1}{3}$

$S_n = \frac{a(r^n - 1)}{r-1} \qquad a = \frac{4}{3}, \ r = 3, \ n = 8$

$\therefore \quad S_8 = \frac{\frac{4}{3}(3^8 - 1)}{3-1}$

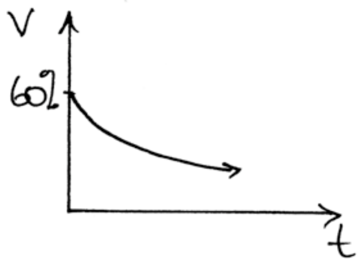
$= \frac{2}{3}(3^8 - 1)$

$= 2(3^7) - \frac{2}{3}$

$= 4373\frac{1}{3}$

## Question 2:

a) i)




ii) $\frac{dV}{dt} < 0$ and $\frac{d^2V}{dt^2} > 0$

b) i) $\alpha$) $x < 0$ or $x > 4$

$\beta$) $x < 2$

$\gamma$) $x = 0, 4$

$\delta$) $x = 2$

$\varepsilon$) $2 < x < 4$

ii) minimum value occurs at $x = -3$ and is $-2$

ie minimum value $= -2$.

c) $y = x^3 + 3x^2 - 9x - 22$

$\frac{dy}{dx} = 3x^2 + 6x - 9$

$\frac{d^2y}{dx^2} = 6x + 6$

for concave up, $\frac{d^2y}{dx^2} > 0$

$\therefore 6x + 6 > 0$

$6x > -6$

$x > -1$.

d) $f(x) = (x^2-3)^4$

$f'(x) = 4(x^2-3)^3(2x)$

$= 8x(x^2-3)^3$

$f''(x) = 8x[3(x^2-3)^2(2x)] + (x^2-3)^3(8)$

$= 8(x^2-3)^2[6x^2 + x^2 - 3]$

$= 8(x^2-3)^2(7x^2-3)$

$\therefore f''(2) = 8(2^2-3)^2(7(2)^2-3)$

$= 200$

Question 3:

a) $y = 4x^3 - x^4$

i) $x$-intercepts occur if $y=0$

$\therefore 0 = 4x^3 - x^4$

$0 = x^3(4-x)$

$x = 0, 4$ ie: at $(0,0)$ and $(4,0)$

ii) $\frac{dy}{dx} = 12x^2 - 4x^3$

$\frac{d^2y}{dx^2} = 24x - 12x^2$

$= 12x(2-x)$

stat. points occur if $\frac{dy}{dx} = 0$

$\therefore 12x^2 - 4x^3 = 0$

$4x^2(3-x) = 0$

$x = 0, 3$

At $x=0$: $y=0$ and

$\frac{d^2y}{dx^2} = 24(0) - 12(0)^2$

$= 0$

$\therefore (0,0)$ is a possible point of inflexion

Checking concavity change:

| $x$ | $0^-$ | $0$ | $0^+$ |
|---|---|---|---|
| $\frac{d^2y}{dx^2}$ | $-$ | $0$ | $+$ |

$\therefore$ concavity changes

$\therefore (0,0)$ is a horizontal point of inflexion

At $x=3$: $y = 4(3)^3 - 3^4$

$= 27$

$\frac{d^2y}{dx^2} = 24(3) - 12(3)^2$

$< 0$ $\cap$

$\therefore (3,27)$ is a local maximum

iii) Inflexions occur if $\frac{d^2y}{dx^2} = 0$

and the concavity changes

$\therefore 12x(2-x) = 0$

$x = 0, 2$

We have already checked $(0,0)$

At $x=2$: $y = 4(2)^3 - 2^4$

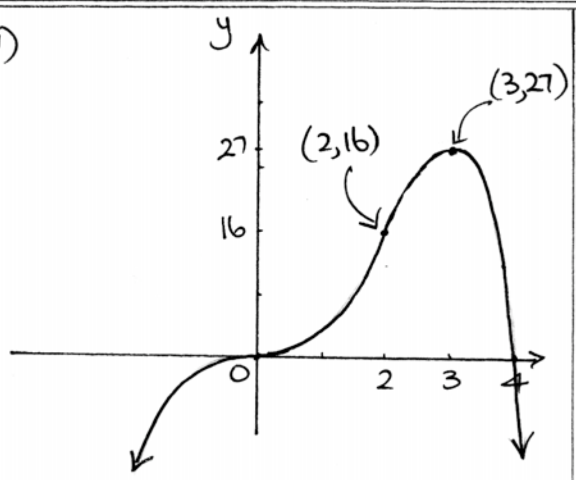
$= 16$

| $x$ | $2^-$ | $2$ | $2^+$ |
|---|---|---|---|
| $\frac{d^2y}{dx^2}$ | $+$ | $0$ | $-$ |

$\therefore$ concavity changes

ie both $(0,0)$ and $(2,16)$ are points of inflexion.

iv)




b) $\dfrac{d^2y}{dx^2} = 6x - 4$ ; stat pt at $(1,12)$

$\dfrac{dy}{dx} = \dfrac{6x^2}{2} - 4x + C$

$= 3x^2 - 4x + C$

subst. $\dfrac{dy}{dx} = 0$ and $x = 1$

$\therefore \; 0 = 3(1)^2 - 4(1) + C$

$0 = 3 - 4 + C$

$C = 1$

$\therefore \dfrac{dy}{dx} = 3x^2 - 4x + 1$

$y = \dfrac{3x^3}{3} - \dfrac{4x^2}{2} + x + d$

$= x^3 - 2x^2 + x + d$

subst. $x = 1,\ y = 12$

$12 = 1^3 - 2(1)^2 + 1 + d$

$d = 12$

$\therefore y = x^3 - 2x^2 + x + 12$

Question 4:

a) i) let $f'(x) = \dfrac{8 + x^2}{x^2}$

$= 8x^{-2} + 1$

$\therefore f(x) = \dfrac{8x^{-1}}{-1} + x + c$

ie $f(x) = -\dfrac{8}{x} + x + c$

ii) let $f'(x) = 4\sqrt[3]{x}$

$= 4x^{1/3}$

$\therefore f(x) = 4 \cdot \dfrac{3}{4} \cdot x^{4/3} + c$

$= 3x^{4/3} + c$

$= 3\sqrt[3]{x^4} + c$

OR $= 3x\sqrt[3]{x} + c$

b) i) $P = 376$ and

$P = 2(x + a + y)$

$\therefore 376 = 2(x + a + y)$

$188 = x + a + y$

$a = 188 - x - y$

as required.

ii) $A_{green} = 6561$

$\therefore ax = 6561$

$a = \dfrac{6561}{x}$

substituting this into i)

$\dfrac{6561}{x} = 188 - x - y$

$\therefore y = 188 - x - \dfrac{6561}{x}$

as required

iii) for a maximum $y$, we need both $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0$

$y = 188 - x - 6561x^{-1}$

$\therefore \dfrac{dy}{dx} = -1 + 6561x^{-2}$

$$\frac{d^2y}{dx^2} = -2(6561)x^{-3}$$

for max/min, $\frac{dy}{dx} = 0$

$$\therefore -1 + 6561x^{-2} = 0$$

$$\frac{6561}{x^2} = 1$$

$$x^2 = 6561$$

$$x = \pm 81$$

but $x > 0$ as $x$ is a length

$\therefore \quad x = 81$

then $\frac{d^2y}{dx^2} = -2(6561)81^{-3}$

$< 0 \quad \cap$

$\therefore$ at $x = 81$, a maximum occurs

then $y = 188 - 81 - \frac{6561}{81}$

$= 26$

and $a = \frac{6561}{81}$

$= 81$

$\therefore \quad a + y = 107$

$\therefore$ the dimensions of the flag are 107 cm by 81 cm

Question 5:

a) i) $6(x-1) + 12(x-1)^2 + 24(x-1)^3 + \ldots$

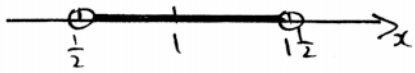
$$r = \frac{12(x-1)^2}{6(x-1)} = \frac{24(x-1)^3}{12(x-1)^2}$$

ie $r = 2(x-1)$

for a limiting sum to exist, $|r| < 1$

$\therefore \quad |2(x-1)| < 1$

$\therefore \quad |x-1| < \frac{1}{2}$




ie $\frac{1}{2} < x < 1\frac{1}{2}$

ii) $S = \frac{a}{1-r}$ ; $|r| < 1$

Now $a = 6(x-1)$ and $x = 1.25$

$\therefore \quad a = 6(1.25 - 1)$

$= \frac{3}{2}$

$r = 2(x-1)$

$= 2(1.25-1)$

$= \frac{1}{2}$

$$\therefore \quad S = \frac{\frac{3}{2}}{1 - \frac{1}{2}}$$

$= 3$

$\therefore$ the limiting sum is 3.

b) $6\%$ pa $= \frac{6}{12}\%$ p month

$= 0.005$ per month.

i) $A_1 = 12000(1.005)^{12} - 1000$

$A_2 = A_1(1.005)^{12} - 1000$

$= [12000(1.005)^{12} - 1000](1.005)^{12} - 1000$

$= 12000(1.005)^{24} - 1000(1.005)^{12} - 1000$

$= 12000(1.005)^{24} - 1000[1 + 1.005^{12}]$

ii) $A_3 = A_2(1.005)^{12} - 1000$

$= 12000(1.005)^{36} - 1000(1 + 1.005^{12} + 1.005^{24})$

$\therefore A_n = 12000(1.005)^{12n} - 1000(1 + 1.005^{12} + 1.005^{24} + \dots + 1.005^{12(n-1)})$

Now $1 + 1.005^{12} + 1.005^{24} + \dots + 1.005^{12(n-1)}$

$= \dfrac{a(r^n - 1)}{r-1}$ where $a = 1$, $r = 1.005^{12}$, $n = n$

$= \dfrac{1\left[(1.005^{12})^n - 1\right]}{1.005^{12} - 1}$

$\therefore A_n = 12000(1.005)^{12n} - 1000\left[\dfrac{1.005^{12n} - 1}{1.005^{12} - 1}\right]$ as required.

iii) Scholarships will cease when $A_n = 0$

ie $12\cancel{000}(1.005)^{12n} - 1\cancel{000}\left[\dfrac{1.005^{12n} - 1}{1.005^{12} - 1}\right] = 0$

$12(1.005)^{12n}(1.005^{12} - 1) - 1.005^{12n} + 1 = 0$

$(1.005)^{12n}\left[12(1.005^{12} - 1) - 1\right] = -1$

$(1.005)^{12n} = \dfrac{-1}{12(1.005^{12} - 1) - 1}$

$(1.005)^{12n} = 3.848\dots$

but $(1.005)^{12\times22} = 3.731\dots$

$(1.005)^{12\times23} = 3.96\dots$

$\therefore$ the full scholarship can only be awarded for 22 years.

iv) $A_{23} = 12000(1.005)^{12\times23} - 1000\left[\dfrac{1.005^{12\times23} - 1}{1.005^{12} - 1}\right]$

$= -476.623\dots$

ie the amount of the scholarship falls short by \$476.62..

$\therefore$ final scholarship $= \$1000 - \$476.62$

$= \$523.38.$